The Isometry Classification of Hermitian Forms over Division Algebras

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ABSTRACT

The isometry classification problem occupies a central role in the theory of quadratic and hermitian forms. This article is a survey of results on the problem for quadratic and hermitian forms over a field and also for hermitian and skew-hermitian forms over a noncommutative division algebra with involution. Rather than adopting a very abstract approach, the problems are stated in matrix or linear-algebraic terms. The known solutions depend crucially on the particular field considered, although there are some general results which are mentioned. While many of the results date back a long time, some recent results, especially those on skew-hermitian forms over a quaternion algebra over a number field, are included.

INTRODUCTION

This paper surveys the isometry classification problem for ε-symmetric bilinear forms over a field ($\varepsilon = \pm 1$), ε-hermitian forms over a field with nontrivial involution, and ε-hermitian forms over a noncommutative central division algebra with involution. Most of the results are well known, but are dispersed in the literature, especially those for the noncommutative case. It seems worthwhile therefore to collect these results and write them in an elementary and palatable way. We describe things in a matrix-theoretic framework rather than adopt a very abstract approach.

The layout of the article is as follows. In Section 1 we give all our basic definitions. In Section 2 we discuss the kinds of invariants that crop up in the isometry classification problem. In Section 3 we consider symmetric bilinear forms over a field (which may equivalently be viewed as quadratic forms provided the field does not have characteristic two), and also quickly dispose

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of skew-symmetric forms. The books of O'Meara [38] and Lam [30] are good references on quadratic forms over fields: the former especially for the classical Hasse theory for algebraic number fields, and the latter for the "algebraic theory of quadratic forms." This "algebraic theory" has flourished in recent years, although its roots go back to the 1937 paper of Witt [60]. See also [34], which deals with forms over rings. In Section 4, we deal with hermitian forms over a field with nontrivial involution. See [9], [21], [34, Appendix 2], [35] as references. In Sections 5–7, we deal with noncommutative division algebras with involution. Most of this material has not appeared in any book, and some of the results have been proven very recently. In Section 8, we make some general comments and mention some results that do not exactly fit into the above scheme of things.

Some familiarity with elementary number theory and the classical Hasse theory would be an advantage in reading this survey, since we deal a lot with algebraic number fields, especially in Sections 5–7. We recommend [38].

1. DEFINITIONS

Let K be a field. Let D be a finite-dimensional central division algebra over K (i.e., D has center K) or else a quadratic extension field of K. Then K may be viewed as a subfield of D in an obvious way. (Note that any division ring is a division algebra over its center, though not necessarily of finite dimension.)

Let $\bar{\ }$ denote an *involution* of D, i.e. an anti-automorphism of period two, i.e.,

$$\overline{x+y}=\bar{x}+\bar{y} \qquad \text{for all} \quad x, y \in D,$$

$$\overline{xy}=\bar{y}\bar{x} \qquad \text{for all} \quad x, y \in D,$$

$$\bar{\bar{x}}=x \qquad \text{for all} \quad x \in D.$$

Let $Z(D)$ denote the center of D. Let $S(D)=\{x \in Z(D): \bar{x}=x\}$, the central symmetric elements.

Then the involution $\bar{\ }$ is said to be of the *first kind* [*second kind*], according as $S(D)=Z(D)$ [$S(D) \neq Z(D)$]; see [1].

If D is noncommutative, then for involutions of either kind we may also refer to the *type* of the involution as being $+1$ [-1] according as $S(D)$ has dimension $n(n+1)/2$ [$n(n-1)/2$], where D has dimension n^2 over K. Note that if $\bar{\ }$ is an involution of type $+1$, then the map $D \to D$, $x \to a^{-1}\bar{x}a$, where $\bar{a}=-a$, will be an involution of type -1, and vice versa.

NOTE. The above definition of type is sensible only when D is noncommutative. If D is a field, then an involution on D is either the identity map or else is conjugation, D being a quadratic extension field of $S(D)$, the fixed field of the involution.

An *ε-hermitian form*, $\varepsilon = \pm 1$, over $(D, ^-)$ is a mapping $\phi: V \times V \to D$, V being a finite-dimensional right D-vector space, such that

$$\phi(x_1 + x_2, y) = \phi(x_1, y) + \phi(x_2, y) \qquad \text{for all} \quad x_1, x_2, y \in V,$$

$$\phi(xd, y) = \bar{d}\phi(x, y) \qquad \text{for all } x, y \in V, \quad \text{for all } d \in D,$$

$$\phi(y, x) = \varepsilon\,\overline{\phi(x, y)} \qquad \text{for all} \quad x, y \in D.$$

[It is easily deduced that $\phi(x, yd) = \phi(x, y)d$ and that $\phi(x, y_1 + y_2) = \phi(x, y_1) + \phi(x, y_2)$.] If $\varepsilon = +1$ $(\varepsilon = -1)$, our form is called *hermitian* (skew-hermitian) except in the special case when D is a field with identity involution, in which case our form is called *symmetric* (*skew-symmetric*). For characteristic not two, symmetric bilinear forms are also called quadratic forms. It should be noted that the distinction between hermitian and skew-hermitian vanishes whenever D contains a central element t such that $\bar{t} = -t$. If ϕ is skew-hermitian then $t\phi$ is hermitian and vice versa (i.e., we may equivalently view a skew-hermitian form as being hermitian via this transformation). This is notably the case when D is a quadratic extension field $K(\sqrt{a})$ with $\overline{\sqrt{a}} = -\sqrt{a}$, and also if D is a noncommutative division algebra with an involution of the second kind, since the involution must fix a subfield K_0 of K, and K will be a quadratic extension of K_0.

Two ε-hermitian forms ϕ_1, ϕ_2 defined on V_1, V_2 are *isometric* (or *equivalent*) if there exists a D-isomorphism $\gamma: V_1 \to V_2$ such that $\phi_2(\gamma x, \gamma y) = \phi_1(x, y)$ for all $x, y \in V_1$. We write $\phi_1 \simeq \phi_2$, and we refer to the map γ as an *isometry* (or *equivalence*). This clearly defines an equivalence relation on the set of ε-hermitian forms. The isometry classification problem is to find invariants which will completely determine the isometry class of a form. (Ideally we seek a complete set of invariants so that two forms are isometric if and only if the values of the corresponding invariants for the two forms coincide.) The nature of these invariants is discussed in Section 2. A solution of the problem for arbitrary K has never been achieved. The known solutions for fields K seem to depend crucially on the kind of field under consideration. There are some general results, however, as we will see later.

We may translate all of the above into the language of matrix theory as follows: Given an ε-hermitian form $\phi: V \times V \to D$, by choosing a D-basis $\{v_1, v_2, \dots, v_n\}$ for V we obtain an $n \times n$ matrix with entries $\phi(v_i, v_j)$. We say this matrix represents the form ϕ with respect to the chosen basis. Write

$A=(\phi(v_i, v_j))$. A different choice of basis will lead to a different matrix B representing ϕ, and $B=\bar{P}^t A P$, where P is an invertible $n \times n$ matrix. (P is, of course, the matrix of change of basis.) We define an equivalence relation $\sim$ on the set of ε-hermitian matrices (i.e. matrices A such that $\bar{A}^t=\varepsilon A$) by $A \sim B$ if and only if $B=\bar{P}^t A P$ for some invertible matrix P of the same size as A and B. This equivalence relation is usually called *congruence*. It may be effected by a sequence of operations as follows.

Perform a column operation on A and then apply the conjugate row operation, i.e., if the column operation is given by right multiplication by an elementary matrix E, then the conjugate row operation is left multiplication by $\bar{E}^t$. Then $A \sim B$ if and only if A can be transformed into B by a sequence of operations as above.

Now if $\phi_i: V_i \times V_i \to D$, $i=1,2$, are represented by matrices A_i, $i=1,2$, with respect to bases $\mathcal{B}_i$ of V_i, $i=1,2$, and if $\gamma: V_1 \to V_2$ is an isometry of ϕ_1 and ϕ_2, then $\bar{P}^t A_1 P = A_2$, where P is the matrix of γ with respect to the bases $\mathcal{B}_1, \mathcal{B}_2$.

The isometry classification problem for ε-hermitian forms thus amounts to classifying ε-hermitian matrices up to congruence.

We say that the ε-hermitian form $\phi: V \times V \to D$ is *nonsingular* if and only if the map $V \to \mathrm{Hom}_D(V, D)$, $x \to \phi(x,\ )$ is bijective. It is immediate that ϕ is nonsingular if and only if any matrix representing ϕ is nonsingular. Equivalently ϕ is nonsingular if and only if $\phi(x,y)=0$ for all $y \in V$ implies $x=0$. The *orthogonal sum* $\phi_1 \perp \phi_2$ of forms $\phi_i: V_i \times V_i \to D$, $i=1,2$, is the map

$$(V_1 \oplus V_2) \times (V_1 \oplus V_2) \to D,$$

$$(x_1 \oplus x_2, y_1 \oplus y_2) \to \phi_1(x_1, y_1) + \phi_2(x_2, y_2).$$

In matrix terms if ϕ_i is represented by A_i, $i=1,2$, then $\phi_1 \perp \phi_2$ is represented by the block matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

In the isometry problem it suffices to consider nonsingular forms, for the following reason. Given ϕ_i, $i=1,2$, on V_i we define the *radical* of ϕ_i, denoted $\mathrm{rad}\, V_i$, to be $\{x \in V_i : \phi_i(x,y)=0\ \ \forall y \in V_i\}$. It is easily shown then that $V_i = U \oplus \mathrm{rad}\, V_i$ for some subspace U_i where $\phi | U_i$ is nonsingular and ϕ splits into an orthogonal sum of $\phi | U_i$ and the zero form on $\mathrm{rad}\, V_i$. It can be shown then that $\phi_1 \simeq \phi_2$ if and only if $\phi_1 | U_1 \simeq \phi_2 | U_2$. Working with matrices, this statement amounts to the fact that

$$\text{if} \quad \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \quad \text{then} \quad A \sim B,$$

where A and B are $k \times k$ matrices, and

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

are $n \times n$ matrices with $k < n$.

A form ϕ is *hyperbolic* if and only if it has a $2n \times 2n$ matrix representation of the form

$$\begin{pmatrix} 0 & \varepsilon I \\ I & 0 \end{pmatrix},$$

where I is the identity $n \times n$ matrix. Equivalently this means that ϕ is isometric to a sum of two-dimensional forms with matrix $\begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$. A useful fact to note is that for any nonsingular form ϕ, the orthogonal sum $\phi \perp (-\phi)$ is hyperbolic [18]. A matrix proof of this comes by showing that the block matrix $\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$ is reduced to $\begin{pmatrix} 0 & \varepsilon I \\ I & 0 \end{pmatrix}$ via the matrix

$$\begin{pmatrix} I & \frac{1}{2}\varepsilon A^{-1} \\ I & -\frac{1}{2}\varepsilon A^{-1} \end{pmatrix}, \qquad \text{where} \quad \overline{A^t} = \varepsilon A$$

(assuming characteristic $K \neq 2$).

Two results that must be mentioned are the following:

(1) Except in the case $D = K$, $^- = 1$, $\varepsilon = -1$, any nonsingular ε-hermitian form has a diagonal matrix representation. The diagonal entries will lie in $\{z \in D : \bar{z} = \varepsilon z\}$. This can be seen by the following argument. There must exist x such that $\phi(x, x) = a$ for some $a \in D$, $a \neq 0$ (because of nonsingularity), and hence ϕ can be represented by a matrix

$$\begin{pmatrix} a & \varepsilon L^t \\ L & N \end{pmatrix},$$

where L is an $(n-1) \times 1$ matrix, and N is an $(n-1) \times (n-1)$ matrix.

Transforming via

$$\begin{pmatrix} 1 & -\frac{1}{a}\varepsilon\bar{L}^t \\ 0 & \varepsilon I \end{pmatrix},$$

our matrix is equivalent to

$$\begin{pmatrix} a & 0 \\ 0 & -\frac{\varepsilon}{a}L\bar{L}^t + N \end{pmatrix}.$$

The result follows by induction.

(2) The Witt cancellation theorem can be proved for nonsingular forms of all kinds, in characteristic $\neq 2$, except $D = K$, $- = 1$, $\varepsilon = -1$. It says that $\phi_1 \perp \phi_2 \simeq \phi_1 \perp \phi_3$ implies $\phi_2 \simeq \phi_3$ [14, 22, 60]. In matrix terms this amounts to

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \sim \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix} \quad \text{implies} \quad A_2 \sim A_3,$$

where $\bar{A}_i^t = \varepsilon A_i$, $i = 1,2,3$. A matrix proof goes as follows (see [39], [43]): It is enough to prove the theorem in the case when $A_1 = a \in D$, i.e. A_1 is a 1×1 matrix, since all forms can be diagonalized. There exists an $n \times n$ matrix P such that

$$\bar{P}^t \begin{pmatrix} a & 0 \\ 0 & A_2 \end{pmatrix} P = \begin{pmatrix} a & 0 \\ 0 & A_3 \end{pmatrix},$$

A_2 and A_3 being $(n-1)\times(n-1)$ matrices. Writing

$$P = \begin{pmatrix} p & \bar{x}^t \\ y & L \end{pmatrix},$$

where x, y are $(n-1)\times 1$, L is $(n-1)\times(n-1)$, and $p \in D$, we have that

$$\bar{p}ap + \bar{y}^t A_2 y = a,$$

$$\bar{p}a\bar{x}^t + \bar{y}^t A_2 L = 0,$$

$$xa\bar{x}^t + \bar{L}^t A_2 L = A_3.$$

Now write $u=p+1$ provided $p+1\neq 0$. Otherwise take $u=p-1$. It is a routine exercise to verify that the $(n-1)\times(n-1)$ matrix $Q=L-yu^{-1}\bar{x}^t$ does what we want, i.e.

$$\bar{Q}^t A_2 Q = A_3.$$

(In the case $D=K$, $\bar{\ }=1$, $\varepsilon=-1$ this cancellation theorem is also true, but in a trivial way, as we shall see in Section 3. In characteristic two the cancellation theorem is false for the forms we have defined, but it is true for quadratic forms, which differ from symmetric bilinear ones. See [34].) [A quadratic form over K is a map $q: V\to K$ such that $q(\alpha x)=\alpha^2 q(x)\ \forall x\in V,\ \forall \alpha\in K$ and that $b: V\times V\to K$, $b(x,y)=q(x+y)-q(x)-q(y)$ is bilinear. Quadratic forms and symmetric bilinear forms are interchangeable in characteristic $\neq 2$, since $b(x,x)=2q(x)$.]

A form ϕ is said to be *isotropic* if $\phi(x,x)=0$ for some x. Otherwise ϕ is said to be *anisotropic*. For all the kinds of forms that we are considering it is possible to decompose ϕ uniquely into an orthogonal sum $\phi_0\perp\phi_1$ where ϕ_0 is anisotropic and ϕ_1 is hyperbolic [8, 18]. To see that such a decomposition exists, suppose ϕ is isotropic. Then if ϕ has matrix A and the vector v is such that $\phi(v,v)=0$, i.e. $\bar{v}^t A v=0$, then we can construct a matrix W, with v as its first row, in such a way that

$$\bar{W}^t A W=\begin{pmatrix} \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & A_1 \end{pmatrix}, \qquad \text{where} \quad \bar{A}_1^t=\varepsilon A_1.$$

Either A_1 is anisotropic or not. If not, we repeat the process above. It follows that we must eventually obtain

$$A=\begin{pmatrix} \begin{pmatrix} 0 & \varepsilon I \\ I & 0 \end{pmatrix} & 0 \\ 0 & B \end{pmatrix},$$

where I is the identity matrix of some dimension and B is anisotropic. The uniqueness of the decomposition follows from the Witt cancellation theorem.

Two forms ϕ,ϕ' are said to be *Witt equivalent* (or of the same Witt class) if and only if there exist hyperbolic forms h,h' such that $\phi\perp h\simeq\phi'\perp h'$. Because of the Witt cancellation theorem it is clear that two nonsingular forms of the same rank will be isometric if and only if they are Witt equivalent. (Two hyperbolic forms of the same rank are automatically isometric.)

Following Witt [60], we may consider the set of Witt classes of ε-hermitian forms over $(D,-)$ and construct a group denoted $W_\varepsilon(D,-)$ with the group operation of orthogonal sum. The class of hyperbolic forms is the zero element, and the class of $(-\phi)$ is the inverse of ϕ since $\phi \perp (-\phi)$ is hyperbolic. When D is a field, we can use the tensor product to define a product of forms [8] (the matrix will be the Kronecker product) and obtain the *Witt ring* of forms. We will return to this later. We say that the form $\phi: V \times V \to D$ *represents* the element $d \in D$ if $\phi(x, x) = d$ for some $x \in V$.

2. INVARIANTS

Dimension

Given any kind of form $\phi: V \times V \to D$, the simplest invariant of ϕ is the *dimension* of the underlying space V as a D-space (i.e. the size of any matrix representing ϕ).

Rank

Secondly we have the *rank* of ϕ, which we define to be the rank of any matrix representing ϕ. Clearly this does not depend on the choice of matrix. When ϕ is nonsingular the rank equals the dimension.

Discriminant

When $D = K$, $- = 1$, we define the discriminant of ϕ to be the determinant of a matrix representing ϕ. It is an element of K, but is only determined up to squares in K, i.e., it is an element of

$$\frac{\dot{K}}{\dot{K}^2} \cup \{0\},$$

$\dot{K}$ being the nonzero elements in K. It is nonzero if and only if ϕ is nonsingular.

When $D = K$ with nontrivial involution, then the discriminant of ϕ is again defined as the determinant of a matrix representing ϕ, but this time it is only determined up to hermitian squares, i.e. elements of the set $\dot{K}\bar{\dot{K}} = \{x\bar{x}: x \in K\}$. So the discriminant lies in

$$\frac{\dot{K}_0}{\dot{K}\bar{\dot{K}}} \cup \{0\},$$

K_0 being the fixed field of the involution.

In either case, when $D = K$, a modification of the definition of discriminant is sometimes made by multiplying by a factor $(-1)^{n(n-1)/2}$. This is to make the discriminant trivial on hyperbolic forms.

When D is a noncommutative division algebra, we define the *discriminant* of a form ϕ over D to be the reduced norm of any matrix representing ϕ [55]. It will belong to $\frac{\dot{K}}{\dot{K}^2} \cup \{0\}$. One way to compute this reduced norm is as follows: Imbed D in $M_r L$, the ring of $r \times r$ matrices with entries in L, for some Galois extension field L of K which splits D, i.e., $D \otimes_K L = M_r L$. The $n \times n$ matrix of ϕ has entries in D and so the embedding yields a matrix in $M_{nr} L$ whose determinant will be the reduced norm. Its value must lie in K, as it will be invariant under the action of the Galois group of L over K. As above, it may sometimes be convenient to multiply by an appropriate power of -1.

Signature

Consider first the case of a nonsingular symmetric bilinear form over $\mathbb{R}$, the real numbers. It has a diagonal matrix representation, and since the diagonal entries only matter up to multiplication by squares in $\mathbb{R}$, we can ensure that these entries are all ± 1. We define the *signature* of ϕ to be $p - m$, where p (m) is the number of times $+1$ (-1) appears on the diagonal. That this is independent of the particular diagonalization is the well-known law of Sylvester. See for example [22] or [24].

NOTE. Other terminology is also commonly used; i.e., the word *index* is sometimes used for signature, and also the terms index and signature are used to denote the integer m, the number of times -1 appears on the diagonal. Also the word *inertia* is used instead of signature. The signature may be defined when the form is singular by simply taking the signature of the nonsingular part. (Beware also that, nowadays, the word index, or Witt index, is often taken to mean the number of hyperbolic plane summands in the decomposition of the form.)

In a similar fashion to the above, a hermitian form over the complex numbers $\mathbb{C}$, or over the real quaternions $\mathbb{H}$, has a real-valued diagonalization with all entries ± 1, thus leading to a signature. Generally, an ordering on any field leads to a signature, because it enables a nonsingular form ϕ to be decomposed into the orthogonal sum $\phi^+ \perp \phi^-$ where ϕ^+ (ϕ^-) is positive (negative) definite. The difference in dimensions of ϕ^+ and ϕ^- is the signature. Also, for various kinds of forms over an algebraic number field K, i.e. a finite extension of the rational field $\mathbb{Q}$, it might happen that localization at an infinite prime will produce a form of one of the above types, thereby yielding a signature. For the basic facts about primes and valuations, see O'Meara [38, chapters I–III]. Note that he uses the term "prime spot" for what we call simply a "prime."

NOTE. Given a form $\phi: V \times V \to D$, D a central division algebra over a number field K, the localization of ϕ at a prime $\mathfrak{p}$, finite or infinite, is the form denoted $\phi_{\mathfrak{p}}$ given by $\phi_{\mathfrak{p}}: V_{\mathfrak{p}} \times V_{\mathfrak{p}} \to D$, where $V_{\mathfrak{p}} = V \otimes_K K_{\mathfrak{p}}$, $D_{\mathfrak{p}} = D \otimes_k K_{\mathfrak{p}}$, $K_{\mathfrak{p}}$ the completion of K at $\mathfrak{p}$, and

$$\phi_{\mathfrak{p}}(x \otimes \lambda, y \otimes \mu) = \phi(x, y) \otimes \bar{\lambda}\mu.$$

If $v_1, v_2, \ldots, v_n$ is a D-basis for V, then $v_1 \otimes 1, \ldots, v_n \otimes 1$ is a $D_{\mathfrak{p}}$-basis, and so $\phi_{\mathfrak{p}}$ is represented by the same matrix as ϕ except that its entries must be regarded as elements of $D_{\mathfrak{p}}$. Beware here that in the case when D is a noncommutative division algebra over K it might happen that $D_{\mathfrak{p}}$ does not remain a division algebra but becomes a full matrix algebra over $K_{\mathfrak{p}}$. We discuss this more fully later.

Algebra classes

A powerful family of invariants exist as algebra classes belonging to some kind of Brauer group of the base field K. The *Brauer group* $B(K)$ is defined as follows: Consider the set of central simple K-algebras. It is well known by Wedderburn's theorem that any central simple algebra A over K is isomorphic to $M_n(D)$, the ring of $n \times n$ matrices with entries in some division algebra D over K. D is called the skewfield part of A. Two central simple algebras are said to be in the same Brauer class if their skewfield parts are isomorphic. The Brauer group $B(K)$ is the set of Brauer classes with the binary operation induced by the tensor product. It is a standard result that the product of two central simple algebras is again central simple. $B(K)$ is a group, the class of K being the identity element and the inverse of A being the class of its opposite algebra A^{op}. (A^{op} is identical to A as a set but has the opposite multiplication, i.e. for $a, b \in A^{\text{op}}$ we define their product $a * b$ in A^{op} to equal ba, the product in A.) It can be shown that $A \otimes A^{\text{op}}$ is a full matrix algebra over K. See [30], for example, for more details and proofs of the above statements.

Now consider a symmetric bilinear form $\phi: V \times V \to K$ (char $K \neq 2$). There is the *Clifford algebra*, denoted $C(\phi)$, defined in the following way (see [30], [38] for more details): First form the tensor algebra $\sum_{n=0}^{\infty} T^n(V)$, where $T^n(V)$ is the n-fold tensor product of V with itself. [$T^0(V) = K, T^1(V) = V$, $T^2(V) = V \otimes V$, etc.] Factor out the two-sided ideal generated by all elements of the form $x \otimes x - \phi(x, x)1_K$, $x \in V$, 1_K being the identity element of K. This is the Clifford algebra $C(\phi)$.

The algebra $C(\phi)$ is central simple if ϕ is even-dimensional, but need not be if ϕ is odd-dimensional. However $C_0(\phi)$ is central simple in this case. The *Witt invariant* of ϕ is defined as the class in the Brauer group of $C(\phi)$, in even dimension, and of $C_0(\phi)$, in odd dimension. Also, a more sophisticated Brauer group, called the Brauer-Wall group BW(K), may be defined. It consists of equivalence classes of $\mathbb{Z}_2$-graded central simple algebras. $C(\phi)$ is $\mathbb{Z}_2$-graded as above, and its class in BW(K) is called the Clifford invariant of ϕ. See [30] for more details.

The most advantageous invariant of this type is perhaps the Hasse invariant, defined as follows: Let ϕ have diagonalization $\langle a_1, a_2, \ldots, a_n \rangle$, ϕ being a nonsingular symmetric bilinear form over K. The *Hasse invariant* of ϕ is the class in the Brauer group $B(K)$ of the product of quaternion algebras

$$\prod_{i<j} \left(\frac{a_i, a_j}{K} \right).$$

(The quaternion algebra $\left(\frac{a_1, a_2}{K} \right)$ is the four-dimensional K-vector space with basis elements $1, i, j, k$ and multiplication defined by $i^2 = a_1$, $j^2 = a_2$, $ij = -ji = k$, where 1 is the identity element of the field K. The prototype for a quaternion algebra is the algebra of real quanternions $\left(\frac{-1, -1}{\mathbb{R}} \right)$. A quaternion algebra is either a division algebra or else the full matrix algebra $M_2 K$.) It can be shown that the Hasse invariant is well defined and independent of the particular diagonalization chosen [11, 30, 51]. It is very amenable to computation.

For hermitian forms over a field with nontrivial involution a complete isometry classification is possible without resorting to invariants of this nature. For ε-hermitian forms over a noncommutative division algebra D a full Clifford algebra cannot be defined. However, an analogue of the even Clifford algebra can be defined. See [2], [23], [52], [55].

This has been used in [2] and [6] to define a relative invariant in the case of ε-hermitian forms over a noncommutative division algebra with involution of the first kind.

Hilbert Symbols

Let K be a p-adic field. Then K has a unique quaternion division algebra and $B(K)$ is cyclic or order two [38]. We may regard $B(K)$ as the multiplicative group $\{\pm 1\}$ with -1 corresponding to the quaternion division algebra. Given $r, s \in K$, we define the Hilbert symbol (r, s) to be $+1$ (-1) according as the quaternion algebra $\left(\frac{r, s}{K} \right)$ is a full matrix algebra (a division algebra).

Equivalently $(r, s) = 1$ if and only if s is a norm from $K(\sqrt{r})$ and also if and only if $rx^2 + sy^2 = 1$ for some $x, y \in K$ [38, p. 164].

In this situation, the Hasse invariant of a quadratic form $\langle a_1, a_2, \dots, a_n \rangle$ over K may be replaced by the Hasse symbol $\prod_{i<j}(a_i, a_j)$ (a product of Hilbert symbols), an element of $\{\pm 1\}$. See [34, p. 78] for a more general notion of symbol over an arbitrary field.

COMMENT 1. There may be slight variations in the definitions of the above invariants. Also there is the Galois-cohomology viewpoint whereby forms themselves and some of the above invariants have an interpretation in terms of elements of certain Galois cohomology groups [51]. See also [2], [29], [48].

It is also possible to define Stiefel-Whitney classes of a quadratic form [13], which are analogous to the characteristic classes used by topologists. Delzant [13] shows that two quadratic forms over an algebraic number field are isometric if they have the same rank and the same Stiefel-Whitney classes. This statement is false in general [48]. See also [37], [4].

COMMENT 2. When a certain collection of invariants are needed for the isometry classification of a particular kind of form, there may well be certain restrictions on the invariants, i.e., not every combination of invariants can occur from a form.

COMMENT 3. In much of our above discussion we have avoided the characteristic-two case. The special types of invariant necessary there will be discussed later.

3. QUADRATIC FORMS OVER A FIELD K

The Real field $\mathbb{R}$ *(or Any Real Closed Field)*

Two nonsingular real quadratic forms are isometric if and only if they have the same rank r and signature σ. See [22, p. 341] for example.

The only restrictions on what may occur are $|\sigma| \leq r$ and $\sigma = r \pmod 2$.

The Complex Field $\mathbb{C}$

Two nonsingular quadratic forms are isometric if and only if they have the same rank. This is because any nonsingular form can be represented by the identity matrix, each element of $\mathbb{C}$ being a square.

The same statements hold for any quadratically closed field.

Finite Fields (Characteristic $\neq 2$)

Two nonsingular forms are isometric if and only if they have the same rank r and discriminant δ [24, p. 342; 30, p. 44]. ($\frac{\dot{K}}{\dot{K}^2}$ consists of two elements, and any nonsingular two-dimensional form represents all nonzero elements of K.) All of the above holds also if K is a transcendental extension of degree one of an algebraically closed field (except that $\left|\frac{\dot{K}}{\dot{K}^2}\right|$ may be greater than two). In particular it is true for $K = \mathbb{C}(x)$, the field of rational functions in one real variable x, with complex coefficients [30, p. 45].

Local (p-adic) Fields

Two nonsingular quadratic forms are isometric if and only if they have the same rank r, discriminant δ, and Hasse invariant S [30, p. 156; 38, p. 170]. This is indeed true for any field over which each five-dimensional form is isotropic [38]. See also Kaplansky [25], who shows that if K is not formally real (i.e. -1 can be written as a sum of squares) and if K has a unique quaternion division algebra, then every five-dimensional form is isotropic. (Note that a p-adic field has the above two properties.) Any combination of r, δ, and S can occur except in dimension one, where

$$S = \left(\frac{d, -1}{K}\right),$$

and in dimension two when $d = -1 \pmod{\dot{K}^2}$ in which case $S = \left(\frac{d, -1}{K}\right)$. See [30] and also [50], where the following alternative approach to the isometry problem is introduced.

Let K be a field with a valuation v, $\operatorname{char} K \neq 2$. Then the valuation ring $V = \{x \in K, v(x) \geqslant 0\}$ is a local ring with maximal ideal $M = \{x \in K; v(x) \geqslant 1\}$.The quotient field $\frac{V}{M}$ is called the residue class field and is denoted by $\bar{K}$. Let K be complete with respect to the valuation v. (If K is complete and $\bar{K}$ finite, then K is called a *local* field.)

Choosing a uniformizing element $\Pi \in K$, i.e. an element Π such that $v(\Pi) = 1$ (Π generates M), we get that any element of K may be written in the form $u\Pi^i$ where u is a unit in the ring V [$v(u) = 1$], and i is a nonnegative integer. For quadratic forms we are only interested in elements of K up to square class. Hence any quadratic form over K has a diagonalization $\langle u_1, u_2, \ldots, u_n, \Pi u_{n+1}, \ldots, \Pi u_r \rangle$, each u_i being a unit in V. Write $\bar{u}_i$ for the image of u_i under the canonical map

$$V \rightarrow \frac{V}{M} = \bar{K}.$$

We obtain two quadratic forms over $\bar{K}$, namely $\langle \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n \rangle$ and $\langle \bar{u}_{n+1}, \ldots, \bar{u}_r \rangle$. These are called the first and second residue class forms of ϕ. See [50], where it is shown that the isometry class of ϕ is completely determined by the isometry classes of these two residue class forms. See also [30, p. 145]. When K is p-adic, $\bar{K}$ is a finite field and quadratic forms over a finite field are easy to handle. This gives a quick way of obtaining a classification for forms over a p-adic field. One drawback to this method is that it fails in characteristic two.

Function Fields

The residue class forms are also useful in dealing with forms over $K(x)$, the field of rational functions in one variable over a field K (characteristic $\neq 2$). Each monic irreducible polynomial $\Pi(x)$ in the polynomial ring $K[x]$ gives rise to a valuation of $K(x)$. We write $K(x)_\Pi$ for the completion of $K(x)$ with respect to this valuation, and $\overline{K(x)}_\Pi$ for the corresponding residue class field. (Note that

$$\overline{K(x)}_\Pi = \frac{K[x]}{(\Pi(x))},$$

a finite extension of K.)

Milnor [37] shows that there is an isomorphism of Witt groups of quadratic forms

$$W(K(x)) \to W(K) \oplus \sum W(\overline{K(x)}_\Pi)$$

the sum being over all monic irreducible polynomials $\Pi(x)$ in $K[x]$. The mapping into the first factor can be described as follows:

Take a monic linear polynomial $\Pi(x)$ so that $\overline{K(x)}_\Pi = K$, and take the first residue class form. For the mapping into the second factor $\Sigma_\Pi W(\overline{K(x)}_\Pi)$ we take, for each Π, the second residue class form. This effectively reduces the classification of quadratic forms over $K(x)$ to classifying forms over finite extensions of K. See also [30]. Also see below for the case when K is finite so that $K(x)$ is a global field.

Algebraic Number Fields

Let K be a finite extension of the rationals. The key result needed to be classify quadratic forms over K is the Hasse principle [20] (or Hasse-Minkowski principle). This says that two nonsingular quadratic forms over K are isometric if and only if they become isometric at the completion of K for each prime

$\mathfrak{p}$. For a proof of this see [38, p. 189]. A complete set of invariants for a classification up to isometry is the rank r, discriminant δ, Hasse symbol ε at each finite prime $\mathfrak{p}$, and signature σ at each real prime $\mathfrak{p}$. There is the usual restriction on signatures, i.e. $|\sigma| \leq r$, $\sigma = r \pmod 2$. There is a restriction on the Hasse symbols due to the Hilbert reciprocity law. Specifically this says that $\varepsilon = -1$ at a finite and even number of primes.

The Hasse principle is also valid for other global fields (i.e. for finite extensions of a rational function field over a finite field). The same set of invariants is needed except that signatures do not arise [38, p. 189].

Fields of Characteristic Two

Symmetric bilinear forms over a field of characteristic two have not been studied as much as quadratic forms, the latter cropping up more naturally in certain situations, especially in topology.

A good treatment of symmetric bilinear forms in characteristic two is by Milnor [36]. Let K be a field of characteristic two. Then K^2 is a subfield of K, and for any symmetric bilinear form $\phi: V \times V \to K$ we have its *value space*, denoted $q(V)$ defined by $q(V) = \{\phi(x,x): x \in V\}$. We view $q(V)$ as a vector space over K^2. It will turn out to be an important invariant of ϕ. Milnor shows that any ϕ may decompose into a sum $\phi_0 \perp \phi_1 \perp \phi_1 \perp h$ where ϕ_0 is anisotropic, h is hyperbolic. In characteristic two for any form ϕ, ϕ and $-\phi$ coincide, but $\phi \perp \phi$ need not be hyperbolic [the result $\phi \perp (-\phi)$ hyperbolic need not be true in characteristic two]. However, $\phi \perp \phi$ will have trivial Witt class. This is possible because the Witt cancellation theorem fails. The Witt class of ϕ is thus determined by ϕ_0. The isometry class of a nonsingular form ϕ on V is proven by Milnor to be determined by ϕ_0, the value space $q(V)$, and the rank r. It is necessary that

$$r = 2\dim_{K^2} q(V) - \operatorname{rank} \phi_0 + 2h,$$

where $2h$ is the rank of a maximal hyperbolic subspace of V.

We briefly discuss quadratic forms in characteristic two, even though they do not quite fit into our general definition of ε-hermitian form.

For K of characteristic two, a quadratic form over K is a map $q: V \to K$ such that $q(\alpha x) = \alpha^2 q(x)$ for all $x \in V$, $\alpha \in K$, and that $b: V \times V \to K$, $b(x,y) = q(x+y) - q(x) - q(y)$ is K-bilinear. The important invariant here is the Arf invariant [5]. (See [34, Appendix 1]; also [10, p. 55].) It is defined as follows.

Choose a symplectic basis $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ for V, i.e. $b(e_i, f_j) = \delta_{ij}$, and define the *Arf invariant* $c(q) = \sum_{i=1}^n q(e_i)q(f_i) \in \tilde{K}$, where $\tilde{K} = K$ modulo $\{\alpha + \alpha^2;\ \alpha \in K\}$. Note that if K is finite, $\tilde{K}$ has two elements, since there is an additive exact sequence

$$0 \to Z_2 \to K \overset{d}{\to} K, \qquad d(\alpha) = \alpha + \alpha^2.$$

Two nonsingular quadratic forms over $\mathbb{Z}_2$ are isometric if and only if they have the same rank and the same Arf invariant. The Arf invariant cannot be defined if q fails to be nonsingular. (q is nonsingular if and only if the associated bilinear form b is nonsingular. Two quadratic forms q_i on V_i, $i = 1, 2$, are isometric if and only if there exists a K-isomorphism $\gamma: V_1 \to V_2$ such that $q_2 \circ \gamma = q_1$.) For fields of characteristic two in general other invariants may also be needed. Sah [46] gives some results showing that in many cases the invariants needed for an isometry classification are the rank, the Arf invariant, and the Clifford algebra. (K being a perfect field is one such case.)

COMMENT 1. We have concentrated on giving the "classical" solutions of the isometry problem for quadratic forms over real, complex, finite, local, and number fields. While there is no known solution for an *arbitrary* field, the problem has been solved for specific classes of field in various specific ways. We may indeed pose the following problem:

Determine all fields whose quadratic forms are classified by a prescribed invariant or set of invariants. See [15] for results on this. There has been much recent work on quadratic forms over fields, on a wide variety of problems, not just isometry classification. For some results of interest and relation to our problem, we refer the reader to [12], [16], [17], [26], [27], [28], [54]. Generally, the kinds of invariant occurring are those in Section 2, though there are other ideas, e.g. the square class invariant of [49], or a function field associated to a quadratic form [26, 54].

The "algebraic theory of quadratic forms," initiated by the 1937 paper of Witt [60], and developed especially by Pfister [40], concerns itself with the ring structure of the Witt ring of quadratic forms described earlier. In particular, the structure of ideals in the Witt ring is well known, and it is interesting to study the connection between the ideal structure of this ring and the kind of invariants needed for an isometry classification. See [30], [34, p. 81]. There are also links with the algebraic K-theory of the field [37]. This algebraic theory of forms can be developed for rings, as instigated by Knebusch [26], and much work has recently been done in this area. In this survey, we have confined ourselves to fields.

COMMENT 2. We must also mention skew-symmetric bilinear forms over a field of characteristic unequal to two. (In characteristic two the distinction between symmetric and skew-symmetric vanishes.) These forms are easily disposed of because it turns out that two such forms are isometric if and only if they have the same rank, which is necessarily even. Any skew-symmetric matrix can easily be shown, via an induction argument, to be congruent to the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. For example, see [22, p. 334].

4. HERMITIAN FORMS OVER A FIELD WITH NONTRIVIAL INVOLUTION

Let L be a field with nontrivial involution. Then L is the quadratic extension $K(i)$, K being the fixed field of the involution. We may take $i^2 = a \in K$, $\bar{i} = -i$ for $\operatorname{char} K \neq 2$, but if $\operatorname{char} K = 2$, we only have $i^2 + i \in K$ and $\bar{i} = 1 + i$.

Note that skew-hermitian forms over L may equivalently be treated as hermitian forms (via the equivalence $\phi \to i\phi$), so we need not consider them separately. A crucial general result is Jacobson's theorem [15], which reduces the isometry classification of hermitian forms over a field to that of quadratic forms. If $\phi: V \times V \to L$ is hermitian, then there is an underlying symmetric bilinear form over the fixed field $K, \psi: V \times V \to K$, given by $\psi(x,y) = \frac{1}{2}[\phi(x,y) + \overline{\phi(x,y)}]$ ($\operatorname{char} K \neq 2$). In characteristic two, there is an underlying quadratic form $q: V \to K$, $q(x) = \phi(x,x)$. In matrix terms, let ϕ be represented by an $n \times n$ matrix $A + iB$, where A, B have entries in K and $A^t = A$, $B^t = -B$. Then ψ is represented by the $2n \times 2n$ matrix

$$\begin{pmatrix} A & aB \\ -aB & -aA \end{pmatrix} \qquad \text{(recall that } i^2 = a\text{)}.$$

Jacobson's theorem says that two hermitian forms over L are isometric if and only if their underlying quadratic forms over K are isometric. In matrices, this says that

$$A_1 + iB_1 \sim A_2 + iB_2 \quad \Leftrightarrow \quad \begin{pmatrix} A_1 & aB_1 \\ -aB_1 & -aA_1 \end{pmatrix} \sim \begin{pmatrix} A_2 & aB_2 \\ -aB_2 & -aA_2 \end{pmatrix},$$

where the congruence on the left is in $(L, \bar{\ })$, and that on the right in K. It would be interesting to see a purely matrix-theoretic proof of this. Jacobson's proof uses an induction argument and the Witt cancellation theorem. Hermitian forms over a field with nontrivial involution turn out to be relatively easy to classify. Observe first that if ϕ has dimension n, then the underlying form has determinant $(-a)^n$.

The Complex Field $\mathbb{C}$

This is the classical case of a hermitian form. Two such forms are isometric if and only if they have the same rank and signature. (The underlying quadratic form will have double the rank and signature.)

Finite Fields

For ϕ hermitian of even dimension the underlying form ψ has determinant one. Hence, by the classification of quadratic forms over a finite field, ϕ must be hyperbolic. For ϕ of odd dimension ψ has determinant $-a$ (mod $\dot{K}^2$), and so it follows that any pair of odd-dimensional forms are isometric. In other words, rank is the only invariant needed for hermitian forms over a finite field. Characteristic two is no different, according to Milnor and Husemoller [34, Appendix 2].

Local (p-adic) Fields

The rank and determinant provide a complete set of invariants for classifying a nonsingular hermitian form ϕ. To see this, note that the underlying form ψ is determined by rank, determinant, and Hasse invariant (see Section 3). If ϕ has a diagonalization $\langle a_1, a_2, \ldots, a_n \rangle$, each $a_i \in K$, then ψ has diagonalization $\langle a_1, a_2, \ldots, a_n, -aa_1, -aa_2, \ldots, -aa_n \rangle$, and it is easily verified that the Hasse invariant of ψ equals $(-a, a_1 a_2 \cdots a_n) = (-a, d)$, where d is the determinant of ϕ (i.e., d determines the Hasse invariant). See also [47].

Algebraic Number Fields

Landherr [31] showed that a complete set of invariants for hermitian forms over a number field with nontrivial involution consists of the rank, the determinant, and a set of signatures, one at each real prime of the field. We can also deduce this from Jacobson's theorem, since, as above, the Hasse invariant of the underlying form ψ will be determined by the determinant d of ϕ. The signatures of ψ at the real primes are clearly twice the corresponding signatures of ϕ.

The usual restrictions on rank and signature apply, and also $d = (-1)^{(r-\sigma)/2}$ at each real prime.

In the case of a function field in one variable over a finite field, the rank and determinant will suffice, there being no signatures in this case.

5. HERMITIAN FORMS OVER A QUATERNION DIVISION ALGEBRA

Let

$$D = \left(\frac{a, b}{K} \right)$$

be the quaternion algebra generated by elements i, j subject to relations $i^2 = a$, $j^2 = b$, $ij = -ji$, etc. We will only be interested in the case when D is a division algebra (i.e. when $ax^2 + by^2 = 1$ is insoluble in K). The standard involution $-$ on D is given by $\bar{i} = -i$, $\bar{j} = -j$. There is also the involution $\wedge$ on D given by $\hat{i} = -i$, $\hat{j} = j$. Both $-$ and $\wedge$ are of the first kind but are of types -1 and $+1$ respectively (see Section 1 for these definitions) since $-$ ($\wedge$) fixes a one-dimensional (three-dimensional) subspace of D. Note that $\hat{x} = i^{-1}\bar{x}i$ for $x \in D$. (Any two involutions on D must differ by an inner automorphism because of the Skolem-Noether theorem [1].) Observe also that if ϕ is ε-hermitian over $(D, \wedge)$ [i.e. $\phi(y, x) = \varepsilon\phi(x, y)$], then $i\phi$ will be $(-\varepsilon)$-hermitian over $(D, \wedge)$. Hence we may, at our convenience, switch from viewing forms as ε-hermitian over $(D, -)$ to $(-\varepsilon)$-hermitian over $(D, \wedge)$.

We first note that Jacobson's theorem [21] is also valid for hermitian forms over $(D, -)$, the underlying form being $\frac{1}{2}(\phi + \bar{\phi})$, as before, in characteristic $\neq 2$, and $\phi(x, x)$ giving an underlying quadratic form in characteristic 2. Thus hermitian forms over $(D, -)$ can be classified if quadratic forms over K can. Skew-hermitian forms, however, have caused much more difficulty, as we shall see later. (Note that we cannot do the trick of changing ϕ to $i\phi$ and regarding skew-hermitian as equivalent to hermitian, because i is not central in D.)

We now describe the situation for hermitian forms over $(D, -)$ for various kinds of field. Note that any such form has a diagonalization with entries in K (see Section 1). It is thus useful to know which elements of K are norms from D, since two one-dimensional forms will be isometric if and only if they differ by a norm from D.

There do not exist any quaternion division algebras over finite fields [1], the complex field, or any quadratically closed field.

The Real Field $\mathbb{R}$

There is unique quaternion division algebra

$$\mathbb{H} = \left(\frac{-1, -1}{\mathbb{R}} \right),$$

the real quaternions. Hermitian forms over ($\mathbb{H}$, $-$) are determined up to isometry by rank and signature, any hermitian matrix $A = \bar{A}^t$ being congruent to a diagonal matrix with all entries ± 1. (All positive reals are norms from $\mathbb{H}$.)

The Local (p-adic) Fields

There is a unique quaternion division algebra D over any p-adic field [38, p. 165]. Given $x \in K$, there exists $z \in D$ such that $z\bar{z} = x$, i.e., any element of K is a norm from D [59, p. 195]. Hence any hermitian matrix over D is congruent to the identity matrix. Two hermitian forms over $(D,-)$ are isometric if and only if they have the same rank.

Algebraic Number Fields

For a given number field K many different quaternion division algebras D may exist. The Hasse principle holds for hermitian forms over $(D,-)$, and so the local invariants will determine the form up to isometry.

At a prime $\mathfrak{p}$ of K, the algebra $D_\mathfrak{p} = D \otimes_K K_\mathfrak{p}$, $K_\mathfrak{p}$ the completion of K at $\mathfrak{p}$, splits at almost all $\mathfrak{p}$, i.e., $D_\mathfrak{p}$ is a full matrix algebra at almost all $\mathfrak{p}$. $D_\mathfrak{p}$ will be a division algebra at a finite number of primes $\mathfrak{p}$, [38]. At a real prime where $D_\mathfrak{p}$ is unsplit we have $D_\mathfrak{p} \cong \mathbb{H}$, and our form has signature at this prime. At all other primes, the only invariant needed is rank, since when $D_\mathfrak{p}$ is split, our form will behave like a skew-symmetric bilinear form, and at a finite unsplit prime, our form is classified by rank alone (see above). Thus a complete set of invariants is the rank together with a set of signatures, one at each real prime $\mathfrak{p}$ where $D_\mathfrak{p}$ is unsplit. The usual relations between rank and signatures are necessary; see [43].

Incidentally, $x \in K$ is a norm from D if and only if x is positive in $K_\mathfrak{p}$ at each real prime $\mathfrak{p}$ where $D_\mathfrak{p}$ is unsplit [59, p. 206].

Function Fields

For a function field K in one variable over a finite field, the completion at a prime $\mathfrak{p}$ is a field of formal power series. Each element of $K_\mathfrak{p}$ is a norm from $D_\mathfrak{p}$, and so two nonsingular hermitian forms of the same rank are isometric; see [43].

6. SKEW-HERMITIAN FORMS OVER A QUATERNION DIVISION ALGEBRA

Let D be a quaternion division algebra with the standard involution $-$. A skew-hermitian form over $(D,-)$ has a diagonalization, the entries lying in $z \in D$; $\bar{z} = -z$. See Pollak [41] for a discussion of the equation $\bar{y} z_1 y = z_2$ in D.

The Real Field

Let $D = \mathbb{H}$, the real quaternions. It may be easily verified that given $z_i \in \mathbb{H}$, $\bar{z}_i = -z_i$, $i = 1,2$, there exists $y \in \mathbb{H}$ such that $\bar{y}z_1y = z_2$. Any nonsingular skew-hermitian matrix over R is thus congruent to the identity matrix, i.e., two nonsingular skew-hermitian forms are isometric if and only if they have the same rank.

The p-adic Fields

These have been studied in [53] and also [43]. Given $z_i \in D$, $\bar{z}_i = -z_i$, $i = 1,2$ (D the unique quaternion division algebra), we can find $y \in D$ such that $\bar{y}z_1y = z_2$ if and only if z_1 and z_2 have the same norm, modulo squares in K. The proof of this needs class field theory. It can then be shown that any four-dimensional skew-hermitian form over $(D,-)$ is isotropic, i.e. represents zero. An induction argument then shows that two nonsingular skew-hermitian forms over $(D,-)$ are isometric if and only if they have the same rank and discriminant. See [53] for details of all this.

There is an alternative approach using residue class forms, as mentioned earlier for quadratic forms over local fields. See Scharlau [47], who shows that, under suitable circumstances, two residue class forms may be defined, and two skew-hermitian forms will be isometric if and only if their residue class forms are isometric.

Algebraic Number Fields

Skew-hermitian forms over a quaternion division algebra over a number field have proved extremely difficult to classify. The problem is that the Hasse principle fails. Two such forms can be isometric at all primes $\mathfrak{p}$ without being globally isometric. We have invariants such as the rank, the discriminant, and a set of signatures at the real prime $\mathfrak{p}$ where $D_{\mathfrak{p}} = M_2\mathbb{R}$. (The form $\phi_{\mathfrak{p}}$ at such real primes behaves like a real quadratic form and gives rise to a signature.) These invariants suffice to determine the form up to isometry locally but not globally, as the Hasse principle fails [29, p. 138]. We thus must try to extend this set of invariants somehow. There are two possible approaches, seemingly different but in fact related, both of which give a procedure for determining whether or not two given forms are isometric. Both approaches are somewhat unsatisfactory insofar as that they do not give a complete set of invariants in the way we have seen for other kinds of form. They both essentially involve a *relative* invariant, i.e. an invariant defined for a pair of forms. The first method is due to Bartels [6, 7] and invokes the theory of algebraic groups and Galois cohomology. The second method [32] is more elementary in that it does not require knowledge of Galois cohomology or algebraic groups but uses exact sequences of Witt groups. The technical details cannot be adequately covered in this survey, and so we refer the reader to [6], [7], and [32].

For other kinds of field which admit a quaternion division algebra we are not sure what results, if any, are known on skew-hermitian forms. If K is a rational function field over a finite field, then similar considerations can be made to those in the number-field case.

7. OTHER NONCOMMUTATIVE DIVISION ALGEBRAS

We exclude at once finite fields and the complex field, as noncommutative division algebras do not exist over these fields. For the real field we have dealt with the only division algebra, the real quaternions. We can also dismiss p-adic fields, since any division algebra over a p-adic field is a cyclic algebra [1, p. 143] (we explain below what is meant by a cyclic algebra), and will not admit an involution unless it is a quaternion algebra [55, p. 125]. These we have encountered already.

Thus we turn to algebraic number fields, where again all the noncommutative division algebras are cyclic [1, p. 149]. If a cyclic algebra has an involution of the first kind, then it must be a quaternion algebra [1, p. 161]. It is possible, however, to obtain a cyclic algebra of any given degree admitting an involution of the second kind. We describe the construction as follows (see [29, p. 81].

First a cyclic algebra of degree n over K is defined by starting with a cyclic extension field L of K, i.e., the Galois group of L over K is cyclic with generator σ and $\sigma^n=1$. A cyclic algebra D consists of all expressions $\sum_{i=0}^{n-1}x_iu^i$, where $x_i\in L$ for each i, u is a symbol, and multiplication is defined by $u^iu^j=u^{i+j}$, $u^n=b$ for some fixed element $b\in K$, and $ux=\sigma(x)u$ for $x\in L$. The identity element is u^0. To ensure that D is a division algebra b must be suitably chosen. To put an involution $-$ of the second kind on D we first must have a nontrivial involution on K so that K is a quadratic extension $K_0(\alpha)$, $\alpha^2\in K_0$, of the fixed field K_0 of this involution. Take $L=L_0K$, where L_0 is a cyclic extension of K_0 that is of degree n and is disjoint from K. The involution $-$ is then defined by $\bar{u}=u^{-1}$, $\bar{x}=x$ if $x\in L_0$, and $-$ coincides with the given involution on K. Note that since $\bar{\alpha}=-\alpha$, the distinction between hermitian and skew-hermitian forms over $(D,-)$ vanishes, i.e., the transformation $\phi\to\alpha\phi$ enables us to change from one to the other.

The Hasse principle holds for such forms [29, p. 77]. See also [43]. Thus we examine what happens at each prime $\mathfrak{p}$ of K_0.

Write $K_\mathfrak{p}=K\otimes_{k_0}(K_0)_\mathfrak{p}$ and $D_\mathfrak{p}=D\otimes_{K_0}(K_0)_\mathfrak{p}$. Then $K_\mathfrak{p}$ is either a field or a double field $(K_0)_\mathfrak{p})\oplus(K_0)_\mathfrak{p}$. If $K_\mathfrak{p}$ is a double field then a form hermitian over $K_{\mathfrak{p}}$ with respect to the involution exchanging components is determined up to isometry by rank alone [43, 55]. So at such a prime we get no special local invariants.

When $K_{\mathfrak{p}}$ is a field, then by a result of Jacobson, $D_{\mathfrak{p}}$ is isomorphic to the full matrix ring $M_n K_{\mathfrak{p}}$. A hermitian form ϕ over D thus gives locally a form $\phi_{\mathfrak{p}}$ over $M_n K_{\mathfrak{p}}$ which will behave like a hermitian form over $K_{\mathfrak{p}}$ with nontrivial involution. At real primes $\mathfrak{p}$, $K_{\mathfrak{p}} \cong C$, the form $\phi_{\mathfrak{p}}$ is determined by a signature, and at finite primes $\mathfrak{p}$ it is determined by the local value of the discriminant. The following elementary matrix treatment is due to Ramanathan [43].

Imbed D in $M_{2n^2} K_{0\mathfrak{p}}$ via the regular representation over K_0. Each element $a \in D$ is represented by a matrix $M_a \in M_{2n^2} K_{0\mathfrak{p}}$. By a change of basis we can get that $M_a = P^{-1}AP$, where $P \in M_{2n^2} K_{0\mathfrak{p}}$ and A is a diagonal block matrix

$$\begin{pmatrix} A_0 & & 0 \\ & A_0 & \\ 0 & & A_0 \end{pmatrix}$$

with n identical blocks A_0. A_0 is a $2n \times 2n$ matrix and is the image of a in the imbedding $D \to D_{\mathfrak{p}} \cong M_n K_{\mathfrak{p}} \subset M_{2n^2} K_{0\mathfrak{p}}$.

If we look at the regular representation of $K_{\mathfrak{p}}$ over $K_{0\mathfrak{p}}$, then any $b \in K$ is represented by a 2×2 matrix

$$\begin{pmatrix} r & \beta s \\ s & r \end{pmatrix},$$

where $\alpha^2 = \beta$, $b = r + \alpha s$, r, $s \in K_{0\mathfrak{p}}$. Thus we may regard A_0 as an $n \times n$ matrix with entries 2×2 matrices of the form $\begin{pmatrix} r & \beta s \\ s & r \end{pmatrix}$. The involution $^-$ on K thus takes

$$\begin{pmatrix} r & \beta s \\ s & r \end{pmatrix} \text{ to } \begin{pmatrix} r & -\beta s \\ -s & r \end{pmatrix}.$$

Hence we define an involution $*$ in the matrix of regular representations by

$$M_a^* = P^{-1}A^*P, \qquad \text{where } A^* = \begin{pmatrix} A_0^* & & 0 \\ & A_0^* & \\ 0 & & A_0^* \end{pmatrix},$$

$A_0^* = \bar{A}_0^t$, A_0 having entries of the form $\begin{pmatrix} r & \beta s \\ s & r \end{pmatrix}$, so that $\bar{A}_0$ has an obvious meaning.

Let us also use $-$ to denote the extension of $-$ on D to an involution on $D_\mathfrak{p}$. It follows that $-$ and $*$ on $D_\mathfrak{p} \subset M_{2n^2} K_{0\mathfrak{p}}$ differ by an inner automorphism, i.e.,

$$\overline{M_a} = Q^{-1} M_a^* Q, \qquad \text{where} \quad Q^* = \pm Q$$

[we can always take $Q^* = Q$, since if $Q^* = -Q$ then $(\alpha Q)^* = \alpha Q$]. Further, $Q = M_b$ for some $b \in D$. Thus

$$Q = M_b = P^{-1} B P \qquad \text{where} \quad B = \begin{pmatrix} B_0 & & 0 \\ & B_0 & \\ 0 & & B_0 \end{pmatrix}.$$

It follows that $(B_0 A_0)^* = B_0 A_0$, i.e., $B_0 A_0$ is the matrix of an n-dimensional hermitian form over $K_\mathfrak{p}$.

If $K_\mathfrak{p} \cong \mathbf{C}$ this matrix has a signature, and if $K_\mathfrak{p}$ is p-adic then the determinant of $B_0 A_0$ may be regarded as a symbol ε taking values ± 1. (The determinant is an element of $\dfrac{K_{0\mathfrak{p}}}{N(K_\mathfrak{p})}$, the hermitian squares, and this set has exactly two elements [38, p. 165]. If the determinant is d, we take $\varepsilon = (\beta, d)$, where $K = K_0(\sqrt{\beta})$.)

Thus two hermitian forms over D will be isometric if and only if they have the same rank, the same signatures at the real primes, and the same symbols at finite primes.

The above process is equivalent to taking a hermitian Morita equivalence of $D_\mathfrak{p}$ to $K_\mathfrak{p}$ [18, 33].

Ramanathan shows that any combination of the above invariants may occur, subject to the usual restrictions on rank and signatures, and to a Hilbert reciprocity law for symbols.

For a function field in one variable over a finite field, Ramanathan says that the same considerations apply except that there will be no signatures.

8. MISCELLANEOUS COMMENTS

Any division ring may be viewed as a division algebra over its center, so the above results are applicable to any division ring finite-dimensional over its center.

Certain other classification problems can be reduced to the classification of ε-hermitian forms over division algebras. For example, the isometry classification of hermitian forms over semi-simple algebras with involution can essentially be so reduced. See [55], [33], where the idea is to first reduce to simple algebras and then, via Morita theory, to division algebras.

The classification of bilinear forms in general over a field, also of sesquilinear forms, is reduced [44, 45] to classifying a set of hermitian forms and quadratic forms associated with a given bilinear of sesquilinear form; see also [57].

The isometry classification of a *pair of quadratic forms* has been studied by Waterhouse [58]. The pairs (ϕ, ϕ') and (ψ, ψ') are *isometric as pairs* if there exists a common isometry γ of ϕ with ψ and of ϕ' with ψ'. If matrices A and B represent ϕ and ϕ', then the idea in [58] is to view $A-\lambda B$ as a form over $K(\lambda)$, a function field. The pair is called a nonsingular pair if $A-\lambda B$ is nonsingular over $K(\lambda)$. To each nonsingular pair there is associated a set of nonsingular quadratic forms over the residue class fields for different valuations of $K(\lambda)$. These determine the pair up to isometry. One particular result is that, for K a number field, the Hasse principle holds for isometry of pairs of forms.

It should be pointed out that in our situation of division algebras with involution, the study of μ-hermitian forms over D [i.e. the forms ϕ with $\phi(y,x)=\mu\phi(x,y)$, where μ is in the center of D] is no more general than the special cases $\mu=\pm 1$. If $^-$ is an involution of the first kind, then we get $\mu\bar{\mu}=1$ which implies $\mu^2=1$ and hence $\mu=\pm 1$. If $^-$ is of the second kind, then $\mu\bar{\mu}=1$ and so, by Hilbert's Theorem 90, $\mu=d^{-1}\bar{d}$ for some $d\in D$. Replacing ϕ by $d\phi$ gives us a hermitian form with respect to the involution $x\to \bar{d}\bar{x}\bar{d}^{-1}$. So if $^-$ is of the second kind, we can take $\mu=1$ after changing the involution.

People sometimes consider the weaker notion of multiplicative equivalence of forms. A *multiplicative equivalence* (or *similarity*) of ε-hermitian forms $\phi_1: V_i\times V_i\to D$, $i=1,2$, is a D-isomorphism $\gamma: V_1\to V_2$ such that $\phi_2(\gamma x,\gamma y)=\lambda\phi_1(x,y)$ for all $x,y\in V_1$ for some fixed $\lambda\in K$, $\bar{\lambda}=\lambda$. If you can solve the usual isometry problem, then this problem is easily resolved.

Also, we have refrained from discussion of another important problem related to the isometry problem, namely the representation problem, i.e., given $d\in D$, does there exist x such that $\phi(x,x)=d$? Many of our references contain lots of information on this problem.

It is possible to have a very general notion of quadratic form (e.g. as in [52] or [56]). See also [42]. However, we have chosen a concrete approach to the subject.

All our forms have been defined on *finite*-dimensional vector spaces. See [19] for recent work on forms on infinite-dimensional spaces.

We have also avoided discussion of the isometry classification problem over rings of various kinds. Much progress has been made in this direction in recent years, but this is outside the ambit of our survey.

Finally, we must admit to not having attempted to trace back a few of the cited results to their earliest appearance. The origins of the theory of quadratic and hermitian forms lie in the last century, if not earlier. See Bourbaki [8] for some early references.

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